Lesson 22: Limits at Infinity

Ex 1 $\lim_{x\to\infty} \frac{4}{x} = 0$

x	10	100	1000	$\to \infty$
$4/x$	.4	.04	.004	$\to 0$

Ex 2 $\lim_{x\to\infty} \frac{x^2+1}{2x^2+1} = \frac{1}{2}$

x	10	100	1000	$\to \infty$
$\frac{x^2+1}{2x^2+1}$	.5025	.5000	.5000	$\to 1/2$

$\frac{101}{201}$

Faster method:
For rational functions $\left(\frac{\text{poly}}{\text{poly}}\right)$, we can look at the limit of the leading terms:

Ex 2(b) $\lim_{x\to\infty} \frac{x^2+1}{2x^2+1} = \lim_{x\to\infty} \frac{\cancel{x^2}}{2\cancel{x^2}} = \lim_{x\to\infty} \frac{1}{2} = \boxed{\frac{1}{2}}$

Ex 3 $\lim_{x\to-\infty} \frac{7x^2+14x+16}{6x-13} = \lim_{x\to-\infty} \frac{7x^{\cancel{2}}}{6\cancel{x}} = \lim_{x\to-\infty} \frac{7x}{6} = \frac{7}{6}(-\text{BIG}) = \boxed{-\infty}$

Recall: A function $f(x)$ has a vertical asymptote $x=c$ if
$\lim_{x\to c^-} f(x) = \pm\infty$ or $\lim_{x\to c^+} f(x) = \pm\infty$.

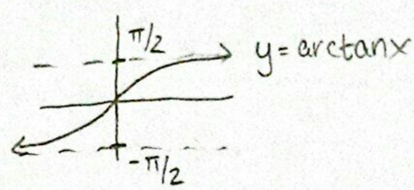
For rational functions, this occurs when we divide by zero $\left(\frac{b}{0}, b \neq 0\right)$.

horizontal line

Def A function $f(x)$ has a horizontal asymptote $y=L$ if

(1) $\lim_{x\to\infty} f(x) = L$ OR

(2) $\lim_{x\to-\infty} f(x) = L$



HA: $y=\frac{\pi}{2}, y=-\frac{\pi}{2}$

Ex 4 Find VA's and HA's:

(a) $f(x) = \dfrac{3x^2+3}{1-x^2}$

HAs: $\displaystyle\lim_{x\to\infty} \frac{3x^2+3}{1-x^2} = \lim_{x\to\infty} \frac{3\cancel{x^2}}{-\cancel{x^2}} = \lim_{x\to\infty} -3 = -3$ $\boxed{y=-3}$

$\displaystyle\lim_{x\to-\infty} \frac{3x^2+3}{1-x^2} = \lim_{x\to-\infty} \frac{3\cancel{x^2}}{-\cancel{x^2}} = \lim_{x\to-\infty} -3 = -3$

VAs: $1-x^2=0$

$\boxed{x=\pm 1}$ (numerator $\neq 0$)

(b) $y = \dfrac{x^2-2x+7}{3x^3+24}$

HAs: $\displaystyle\lim_{x\to\infty} \frac{x^2-2x+7}{3x^3+24} = \lim_{x\to\infty} \frac{\cancel{x^2}}{3x^{\cancel{3}}} = \lim_{x\to\infty} \frac{1}{3x} = 0$

$\displaystyle\lim_{x\to-\infty} \frac{x^2-2x+7}{3x^3+24} = \lim_{x\to-\infty} \frac{\cancel{x^2}}{3x^{\cancel{3}}} = \lim_{x\to-\infty} \frac{1}{3x} = 0$ $\boxed{y=0}$

VAs: $3x^3+24=0$

$3x^3 = -24$

$x^3 = -\frac{24}{3} = -8$

$x = \sqrt[3]{-8} = -2$

$\boxed{x=-2}$

Sometimes we don't get a HA:

(c) $h(x) = \dfrac{8x^3+2x^2+1}{4x^2-16}$

HA: $\displaystyle\lim_{x\to\infty} \frac{8x^3+2x^2+1}{4x^2-16} = \lim_{x\to\infty} \frac{8x^{\cancel{3}}}{4\cancel{x^2}} = \lim_{x\to\infty} \frac{2\cancel{8}x}{\cancel{4}} = \infty$ $\boxed{\text{no HA's}}$

$\displaystyle\lim_{x\to-\infty} \frac{8x^3+2x^2+1}{4x^2-16} = \lim_{x\to-\infty} 2x = -\infty$

VA: $4x^2-16=0$

$\boxed{x=\pm 2}$

<u>Def</u> A <u>slant asymptote</u> is a line $y = mx + b$ $(m \neq 0)$ that $f(x)$ approaches as $x \to \infty$ or $x \to -\infty$. These occur if the degree of the numerator (the biggest power of x) is one larger than the degree of the denominator.

we find these using polynomial long division.

<u>Ex 4(c)</u>

$$h(x) = \frac{8x^3 + 2x^2 + 1}{4x^2 - 16} \quad \begin{array}{l} \leftarrow \text{deg } 3 \\ \leftarrow \text{deg } 2 \end{array}$$

so we have a slant asymptote

$$\begin{array}{r l}
 & 2x + \tfrac{1}{2} \qquad R\ -32x+9 \\
4x^2 - 16 \,\big)\!\! & \overline{8x^3 + 2x^2 + 0x + 1} \\
 & -(8x^3 \qquad\quad - 32x) \\
\hline
 & 2x^2 - 32x + 1 \\
 & -(2x^2 \qquad\quad - 8) \\
\hline
 & -32x + 9
\end{array}$$

warmup 2:

$$\begin{array}{r l}
 & 28 \ R\ 3 \\
31 \,\big)\!\! & \overline{871} \\
 & -62\downarrow \\
\hline
 & 251 \\
 & -248 \\
\hline
 & 3
\end{array}$$

$$\frac{871}{31} = 28 + \frac{3}{871}$$

$$\frac{8x^3 + 2x^2 + 1}{4x^2 - 16} = \underbrace{2x + \tfrac{1}{2}} + \frac{-32x + 9}{4x^2 - 16}$$

$$\boxed{y = 2x + \tfrac{1}{2}}$$